ON NANO SOFT $^s(\mathscr{I})$-CONTINUOUS FUNCTIONS

S. P. R. Priyalatha, S. Vanitha* and T. Witczak**

Department of Mathematics,
Kongunadu Arts and Science College,
Coimbatore - 641029, Tamil Nadu, INDIA

E-mail : priyalathamax@gmail.com

*Department of Mathematics,
Bharathiyar Institute of Engineering for Women,
Salem - 636112, Tamil Nadu, INDIA

E-mail : svanithamaths@gmail.com

**Institute of Mathematics,
University of Silesia,
Bankowa 14, Katowice, POLAND

E-mail : tm.witczak@gmail.com

Abstract: In this paper, we introduce new ideas of nano soft $^s(\mathscr{I})$-open sets and examine a few characteristics of nano soft $^s(\mathscr{I})$-openess. Furthermore, we study nano soft $^s(\mathscr{I})$-continuous functions. Likewise, some characterization and properties are given. We compare our framework with some other well-known ones that are present in the literature.

Keywords and Phrases: Nano soft $^s(\mathscr{I})$-open, nano soft $^s(\mathscr{I})$-closed set, nano soft $^s(\mathscr{I})$-continuity.

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1. Introduction

In this paper we combine several mathematical tools which are all connected with general topology or with the ideas of uncertainty and approximation. Due to this reason, we would like to present briefly some basic components of our project.

These are: soft sets (together with soft topologies), nano topologies (with rough sets) and ideals (with ideal topologies).

Soft set theory was introduced by Molodtsov in 1999. The importance of this theory lies in the fact that it is a serious generalization of many other theories modelling various aspects of uncertainty. As an illustration, fuzzy sets may be thought of as a particular instance of soft sets.

The past two decades brought fast development of this branch of mathematics. For example, Shabir and Naz defined and studied soft topological spaces(TS) in [23]. Independent definitions of soft topologies were presented e.g. by Çagman in [5] or by Roy and Samanta in [21]. Many other authors contributed to this area.

As for the *nano topological* spaces, it seems that their study was initiated by Lellis Thivagar in 2013 (see [13]). Since then, various mathematicians proved many important results on nano-topologies. There are also practical applications of these systems (e.g. in medical diagnostics). However, the whole concept relies on the older idea of rough sets which was developed by Pawlak already in 80s (see [19]). Rough sets can be considered as approximations of classical sets (in a sense that will be explained later).

The very concept of *ideal topology* was presented by Kuratowski and Vaidyanatha-swamy. The notion of ideal continuity was introduced by Hashimoto in 1952 (see [9]). *I*-open sets in ideal TS were introduced by Jankovic and Hamlett in [8]. We can say that in this specific context ideals are generalizations of the notion of empty set. This allows to redefine the notion of closure. Instead of ideals, we can use even more general families of sets: arbitrary hereditary classes.

The hypothesis of soft *I*-open set and soft *I*-continuity of functions was originated by Akdag and Erol in [3].

The aim of the present paper is to establish the idea of nano soft $^s(\mathscr{I})$-open sets and nano soft $^s(\mathscr{I})$-continuous functions. Our approach is different (at least to certain extent) than the one presented e.g. in [3]. For example, we do not use the notion of local function. Our aim is to approximate some important families of classical sets (e.g. ideals) by means of soft sets.

2. Preliminaries

Here, we review a few traditional definitions. Generally speaking, we presumptively know certain fundamental elements of the theories given in the opening chapter (that is, with soft sets, nano-topologies and the notion of ideal). However, we would like to agree on notation and terminology.

2.1. Soft sets

Let us start from the very definition of soft set and its basic operations.

Definition 2.1. [16] *Assume that Y is a non-empty universe and E is a non-empty set of parameters. Let $A \subseteq E$. A mapping $\mathscr{U}_A$ from A to $P(Y)$ is called a soft set on Y.*

As for the examples of soft sets, the reader can find them easily in many papers, e.g. in [4], [17] and [3].

Remark 2.2. *One can see that we can always assume that our soft set is defined for each $e \in E$. If we have $\mathscr{U}_A$ where $A \subseteq E$, $A \neq E$ and there is some $g \in E$ $g \notin A$, then we may agree that $\mathscr{U}(g) = \emptyset$.*

Definition 2.3. [18] *Let Y be a universe, E be a set of parameters and $A \subseteq E$. Then $\mathscr{U}_A$ is called a relative null soft set (or relative empty soft set), represented $\tilde{\emptyset}_A$, if $\mathscr{U}(a) = \emptyset$ for any $a \in A$. It is called absolute null soft set if $A = E$.*

Moreover, $\mathscr{U}_A$ is called a relative whole soft set (or a relative universal soft set), represented by $\tilde{Y}_A$, if $\mathscr{U}(a) = Y$ for any $a \in A$. It is called absolute whole soft set if $A = E$.

Definition 2.4. [16] *Let Y be a universe and E be the set of parameters. Then a soft set $\mathscr{U}_E$ over Y is called a soft point $\{x\}_a$ (or $\tilde{x}_a$) if there is exactly one $a \in E$ $\mathscr{U}(a) = \{x\}$ for some $x \in Y$ and $\mathscr{U}(e) = \emptyset$ for any $e \in E \setminus \{a\}$. If $\mathscr{V}_E$ is another soft set over Y then we can say that $\tilde{x}_a \in \mathscr{V}_E$ iff $x \in \mathscr{V}(a)$.*

Remark 2.5. *There are other definitions of soft points too. They are not necessarily equivalent with the one presented above. The reader can check an interesting survey [22] with some new results.*

Definition 2.6. [16] *Let $\mathscr{U}_A$ and $\mathscr{V}_B$ be two soft sets over a common universe $\tilde{Y}$. Then $\mathscr{U}_A$ is said to be a soft subset of $\mathscr{V}_B$ if $A \subseteq B$ and for all $e \in A$, $\mathscr{U}(e) \subseteq \mathscr{V}(e)$. In this case we write that $\mathscr{U}_A \subseteq \mathscr{V}_B$. Two soft sets are equal if they have the same set of parameters and each parameter is connected with the same subset of Y.*

As for the operations on soft sets, we may speak e.g. about union, intersection and complement. Various approaches exist in literature. The reader is encouraged to check an interesting overview [18] where many existing definitions are compared and analyzed. Another paper is [2] where some additional soft union, intersection and subset operators were investigated.

It is convenient (especially for our purposes) to assume that soft sets are defined for each $e \in A$, as it was presented in Remark 2.2. Then the following definitions are reasonable.

Definition 2.7. [16] *Let $\mathscr{U}_A$ and $\mathscr{V}_B$ be two soft sets defined on the same universe Y with E as the set of parameters. Then we define:*

1. *Soft union of $\mathscr{U}_A$ and $\mathscr{V}_B$ as a new soft set $\mathscr{H}_C$ for any $e \in E$, $\mathscr{H}_C = \mathscr{U}_A(e) \cup \mathscr{V}_B(e)$.*

2. *Soft intersection of $\mathscr{U}_A$ and $\mathscr{V}_B$ as a new soft set $\mathscr{H}_C$ for any $e \in E$, $\mathscr{H}_C(e) = \mathscr{U}_A(e) \cap \mathscr{V}_B(e)$.*

3. *Soft complement of $\mathscr{U}_A$ as $-\mathscr{U}_A$ where for any $e \in E$, $-\mathscr{U}_A(e) = Y \setminus \mathscr{U}_A(e)$.*

As for the soft difference of $\mathscr{U}_A$ and $\mathscr{V}_B$, it can be defined (from the logical point of view) as $\mathscr{U}_A \cap -\mathscr{V}_B$.

Remark 2.8. *Note that we could use some special symbols to denote operations on soft sets (contrary to the operations on classical sets). For example, we could write $\mathscr{U}_A \tilde{\subseteq} \mathscr{V}_B$ or $\mathscr{U}_A \tilde{\cup} \mathscr{U}_B$. However, we think that the risk of confusion is not serious and the context is enough to avoid any misunderstanding. Due to the same reason sometimes we shall omit adjective soft when it is clear that we mean union or intersection of soft sets. Moreover, according to Remark 2.2 we shall often discard subscripts A or E in the denotation of a given soft set (hence we shall write $\mathscr{U}_A$ instead of $\mathscr{U}_E$) when no risk of confusion arises.*

Definition 2.9. [4] *Let $\mathscr{U}_A$ and $\mathscr{V}_B$ be two soft sets over Y. The Cartesian product of $\mathscr{U}_A$ and $\mathscr{V}_B$ is represented as $\mathscr{U}_A \times \mathscr{V}_B$ and defined as an ordered pair $(\mathscr{H}, A \times B)$, where $\mathscr{H} : A \times B \to P(Y \times Y)$ and $\mathscr{H}(a,b) = \mathscr{U}_A(a) \times \mathscr{V}_B(b)$, where $(a,b) \in A \times B$. In other words, for any $(a,b) \in A \times B$, $\mathscr{H}(a,b) = \{(h_i, h_j);$ where $h_i \in \mathscr{U}_A(a)$ and $h_j \in \mathscr{V}_B(b)\}$.*

Definition 2.10. [4] *Let $\mathscr{U}_A$ and $\mathscr{V}_B$ be two soft sets over the same universe Y. Binary soft relation R from $\mathscr{U}_A$ to $\mathscr{V}_B$ is defined as a subset of $\mathscr{U}_A \times \mathscr{V}_B$. In particular, it is possible that $\mathscr{V}_B = \mathscr{U}_A$. In this case we can speak about soft relation on $\mathscr{U}_A$ and we can write that for any $a, b \in A$, $\mathscr{U}_A(a) \, R \, \mathscr{U}_A(b)$ iff $\mathscr{U}_A(a) \times \mathscr{U}_A(b) \in R$.*

Definition 2.11. [4] *Let R be a soft relation on $\mathscr{U}_A$. Then we say that R is:*

(i) *soft reflexive if $\mathscr{U}_A(a) \times \mathscr{F}_A(a) \in R$, $\forall a \in A$.*

(ii) *soft symmetric if $\mathscr{U}_A(a) \times \mathscr{U}_A(b) \in R \Rightarrow \mathscr{U}_A(b) \times \mathscr{U}_A(a) \in R$, $\forall a, b \in A$.*

(iii) *soft transitive if $\mathscr{U}_A(a) \times \mathscr{U}_A(b) \in R, \mathscr{U}_A(b) \times \mathscr{U}_A(c) \in R \Rightarrow \mathscr{U}_A(a) \times \mathscr{U}_A(c) \in R$, $\forall a, b, c \in A$.*

A soft relation on $\mathscr{U}_A$ that is reflexive, symmetric and transitive is called an equivalence relation.

At first glance, the very definition of soft set is rather simple. However, working with soft sets requires some subtle prudence. We should remember that the elements of soft set are classical sets (and these classical sets have their own elements). This will be important in the next definition.

Definition 2.12. [4] *Let $\mathscr{U}_A$ be a soft set and R be an equivalence relation on $\mathscr{U}_A$. Then the soft equivalence class of $\mathscr{U}_A(a)$ is defined as $[\mathscr{U}_A(a)]_R = \{\mathscr{U}_A(b) : \mathscr{U}_A(b)R\mathscr{U}_A(a)\}$.*

The reader can find some examples of soft relations and soft equivalence classes e.g. in [4]. We would like to emphasize the fact that soft equivalence classes are soft sets. In particular, they are soft subsets of $\mathscr{U}_A$. Hence, their elements are not the elements of universe Y as such but rather classical sets with appropriate parameters.

2.2. Soft topology

Having soft sets as such, we may formulate the idea of soft topological space.

Definition 2.13. [23] *Let $SS(Y)$ be the collection of all soft sets over non-empty universe Y. Assume that $\tilde{\iota} \subseteq SS(Y)$. Then $\tilde{\iota}$ is said to be a soft topology on Y if and only if:*

(i) $\tilde{\emptyset}, \tilde{Y} \in \tilde{\iota}$.

(ii) The union of any number of soft sets in $\tilde{\iota}$ belongs to $\tilde{\iota}$.

(iii) The intersection of any two soft sets in $\tilde{\iota}$ belongs to $\tilde{\iota}$.

The triplet $(Y, \tilde{\iota}, E)$ is called soft topological space over Y (where E is the set of parameters). The members of $\tilde{\iota}$ are said to be $\tilde{\iota}$-soft open sets. A soft set over Y is said to be $\tilde{\iota}$-soft closed in Y if its complement belongs to $\tilde{\iota}$.

In the following theorems we assume that any soft set in a given soft topological space is defined for the whole set E of parameters (in the sense of Remark 2.2).

Definition 2.14. [23] *Let $(\tilde{Y}, \tilde{\iota}, E)$ be a soft topological space over Y and $\mathscr{U}_E$ be a soft set over $\tilde{Y}$. The soft closure of $\mathscr{U}_E$ represented by $cl(\mathscr{U}_E)$ is the intersection of all closed soft supersets of $\mathscr{U}_E$. The soft interior of $\mathscr{U}_E$ represented by $int(\mathscr{U}_E)$ is the union of all open soft subsets of $\mathscr{U}_E$.*

Definition 2.15. [15] *Let $(Y, \tilde{\iota}, E)$ be a soft topological space over Y, $\mathscr{V}_E$ be a soft closed set in Y and $\mathscr{E}_E$ be a soft point $\mathscr{E}_E \notin \mathscr{V}_E$. If there exists soft open sets $\mathscr{U}_E$ and $\mathscr{H}_E$ $\mathscr{E} \in \mathscr{U}_E$, $\mathscr{V}_E \subseteq \mathscr{H}_E$ and $\mathscr{U}_E \cap \mathscr{H}_E = \tilde{\emptyset}$ then $(\tilde{Y}, \tilde{\iota})$ is called a soft regular space.*

Definition 2.16. [15] *Let $(Y, \tilde{\iota}, E)$ be a soft topological space over Y, and let $\mathscr{U}_E$ and $\mathscr{V}_E$ be two disjoint soft closed sets. If there exist two soft open sets $\mathscr{M}_E$ and $\mathscr{N}_E$ $\mathscr{U}_E \subseteq \mathscr{M}_E$, $\mathscr{V}_E \subseteq \mathscr{N}_E$ and $\mathscr{M}_E \cap \mathscr{N}_E = \tilde{\emptyset}$ then $(Y, \tilde{\iota})$ is called a soft normal space.*

In the next definition we determine the notion of soft ideal.

Definition 2.17. [11] *Let $\mathcal{X}$ be a non empty collection of soft sets over Y, with the same set of parameters E. Then $\mathcal{X} \subseteq SS(Y)_E$ is called a soft ideal on Y with the same set E if:*

(i) *$\mathscr{U}_E \in \mathcal{X}$ and $\mathscr{V}_E \in \mathcal{X} \Rightarrow \mathscr{U}_E \cup \mathscr{V}_E \in \mathcal{X}$.*

(ii) *$\mathscr{U}_E \in \mathcal{X}$ and $\mathscr{V}_E \subseteq \mathscr{U}_E \Rightarrow \mathscr{V}_E \in \mathcal{X}$, i.e. $\mathcal{X}$ is closed under finite soft unions and soft subsets.*

2.3. Nano topology

In this subsection we recall the very definition of nano-topological space and we reconstruct it in the context of soft sets together with ideals.

Definition 2.18. [13] *Let Y be a non empty finite set of objects called the universe, R be an equivalence relation on Y named indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (Y, R) is said to be an approximation space. Let $X \subseteq Y$.*

(i) *The lower approximation of X with respect to R is the set of all objects that can be for certain classified as members of X with respect to R and it is represented by $L_R(X)$. That is, $L_R(X) = \left\{ \bigcup_{x \in Y} \{R(x) : R(x) \subseteq X\} \right\}$, where $R(x)$ denotes the equivalence class determined by x.*

(ii) *The upper approximation of X with respect to R is the set of all objects that can be possibly classified as members of X with respect to R and it is represented by $U_R(X) = \left\{ \bigcup_{x \in Y} \{R(x) : R(x) \cap X \neq \emptyset\} \right\}$.*

(iii) *The boundary region of X with respect to R is the set of all objects which can be classified neither as in X nor as not in X with respect to R and it is represented by $B_R(X) = U_R(X) \setminus L_R(X)$.*

Following Pawlak, we say that a tuple $(L_R(X), U_R(X))$ is a rough set with a target set X and with respect to R.

We may observe that the idea of approximation space allows us to discover and define certain special topology.

Definition 2.19. [13] *Let Y be the universe, R be an equivalence relation on Y and $X \subseteq Y$. Then $\tilde{\iota}_R(X) = \{Y, \emptyset, L_R(X), U_R(X), B_R(X)\}$ is called nano topology on X. Hence, $\tilde{\iota}_R(X)$ forms a topology on Y.*

As we have already written in Introduction, nowadays there are many papers about nano-topologies and their applications. Moreover, some other concepts have been derived from this initial idea.

Definition 2.20. [20] *Let Y be a non- null set, E be a set of parameters and $A \subseteq E$. Let $\mathscr{U}_A$ be a soft set on Y. Let R be a soft equivalence relation on $\mathscr{U}_A$ (hence the elements of soft R-equivalence class of $\mathscr{U}_A(a)$ are represented by $[\mathscr{U}(a)]$). Let $^s(\mathscr{I})$ be an soft ideal set on Y: we assume tacitly that is defined for some subset of parameters that is contained in A. The ordered pair $(Y, \mathscr{U}_A, {}^s(\mathscr{I}))$ is called soft ideal approximation space. Then the soft ideal L- approximation, U-approximation, boundary region of $^s(\mathscr{I})$ are defined as follows:*

(i) $L_R(^s\mathscr{I}) = \bigcup\limits_{a \in A} \{[\mathscr{U}(a)] : [\mathscr{U}(a)] \subseteq (^s\mathscr{I})\}$.

(ii) $U_R(^s\mathscr{I}) = \bigcup\limits_{a \in A} \{[\mathscr{U}(a)] : [\mathscr{U}(a)] \cap (^s\mathscr{I}) \neq \tilde{\emptyset}\}$.

(iii) $B_R(^s\mathscr{I}) = U_R(^s\mathscr{I}) \setminus L_R(^s\mathscr{I})$.

Let us analyze the following example that is based on the Example 3.4 from [20] but with some adjustments and changes.

Example 2.21. Let $Y = \{s_1, s_2, s_3, s_4, s_5\}$ be our universe and $E = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}$ be the set of parameters. Assume that $A = \{r_1, r_2, r_3, r_4\}$. Clearly, $A \subseteq E$. Now let us define $\mathscr{U}_A = \{(r_1, \{s_1\}), (r_2, \{s_3\}), (r_3, \{s_2, s_4\}), (r_4, \{s_4, s_5\})\}$.

Let:

$^s(\mathscr{I}) = \{(r_1, \{s_1, s_2\}), (r_2, \{s_1, s_2, s_3\}), (r_3, \{s_1, s_2, s_3\}), (r_4, \{s_1\}), (r_5, \{s_2\}), (r_6, \{s_3\}), (r_7, \emptyset), (r_8, \{s_1, s_3\}), (r_9, \{s_2, s_3\})\}$.

Undoubtedly, $\mathscr{U}_A$ is a soft set over Y. For example, $\mathscr{U}(r_3) = \{s_2, s_4\}$ (we may omit A in the subscript). Furthermore, $^s(\mathscr{I})$ is an soft ideal set. When treated just as a family of classical sets, it is closed under unions and subsets (as the reader can easily check). Besides, in classical setting we cannot use $\{s_1, s_2, s_3\}$ (or any other subset) twice (rather, it would be trivial). However, here this set is associated with two different parameters. Hence, this approach is close to the idea of multi-sets.

Now let $R = \{\mathscr{U}(r_1) \times \mathscr{U}(r_1), \mathscr{U}(r_2) \times \mathscr{U}(r_2), \mathscr{U}(r_3) \times \mathscr{U}(r_3), \mathscr{U}(r_4) \times \mathscr{U}(r_4), \mathscr{U}(r_1) \times \mathscr{U}(r_2), \mathscr{U}(r_2) \times \mathscr{U}(r_1)\}$. Then we may calculate equivalence classes:

$[\mathscr{U}(r_1)] = \{\mathscr{U}(r_1), \mathscr{U}(r_2)\}$, $[\mathscr{U}(r_2)] = \{\mathscr{U}(r_1), \mathscr{U}(r_2)\}$, $[\mathscr{U}(r_3)] = \{\mathscr{U}(r_3)\}$, $[\mathscr{U}(r_4)] = \{\mathscr{U}(r_4)\}$. Clearly, each of these classes is a soft set. Now we see that:

$[\mathscr{U}(r_1)] \subseteq {}^s(\mathscr{I})$ (in the sense of soft inclusion). This is because $\mathscr{U}(r_1) \subseteq {}^s(\mathscr{I})(r_1)$ (note that $\{s_1\} \subseteq \{s_1, s_2\}$) and $\mathscr{U}(r_2) \subseteq {}^s(\mathscr{I})(r_2)$ (of course $\{s_3\} \subseteq \{s_1, s_2, s_3\}$). Thus $[\mathscr{U}(r_2) \subseteq {}^s(\mathscr{I})$ too.

On the other hand, one can easily check that $[\mathscr{U}(r_3)] \nsubseteq {}^s(\mathscr{I})$ and $[\mathscr{U}(r_4) \nsubseteq {}^s(\mathscr{I})$. This is because $\mathscr{U}(r_3) = \{s_2, s_4\} \nsubseteq {}^s(\mathscr{I})(r_3) = \{s_1, s_2, s_3\}$ and $\mathscr{U}(r_4) = \{s_2, s_4\}\{s_4, s_5\} \nsubseteq {}^s(\mathscr{I})(r_4) = \{s_1\}$.

Thus we may formulate $L_R({}^s\mathscr{I}) = [\mathscr{U}(r_1)] \cup [\mathscr{U}(r_2)] = [\mathscr{U}(r_1)] = \{(r_1, \{s_1\}), (r_2, \{s_3\})\}$.

Now we see that $\{s_2, s_4\} \cap \{s_1, s_2, s_3\} \neq \emptyset$. Hence, $[\mathscr{U}(r_3)] \cap {}^s(\mathscr{I}) \neq \tilde{\emptyset}$. On the other hand $\{s_4, s_5\} \cap \{s_1\} = \emptyset$. Thus $[\mathscr{U}(r_4)] \cap {}^s(\mathscr{I}) = \tilde{\emptyset}$.

Thus we formulate $U_R({}^s\mathscr{I}) = [\mathscr{U}(r_1)] \cup [\mathscr{U}(r_2)] \cup [\mathscr{U}(r_3)] = \{(r_1, \{s_1\}), (r_2, \{s_3\}), (r_3, \{s_2, s_4\}\}$. Clearly, $\mathscr{B}_R({}^s\mathscr{I}) = \{(r_3, \{s_2, s_3\})\}$.

As the reader could see, in this concept we approximate ideals using soft sets. That is, we approximate certain families of sets using other families (because soft sets are parametrized families of sets, as we know). Clearly, ${}^s(\mathscr{I})$ plays the role of X in the initial definition of rough set and nano topology. The reader is encouraged to compare this approach with the concept of Akdag and Erol (who used soft ideals together with the idea of local function, see [3]) and with the approach presented by Kandil et al. in their paper [12]. Yet another solution was presented by Esmaeel and Mohammad in [7].

Now we have some other definitions.

Definition 2.22. [20] *Let Y be a universe, E be the set of parameters, $A \subseteq E$, $\mathscr{U}_A$ be a soft set over Y and ${}^s(\mathscr{I})$ be an soft ideal set defined for at least those parameters which are in A. Then $(Y, \mathscr{U}_A, {}^s(\mathscr{I}))$ is a ordered triplet of soft ideal approximation space and $\tilde{\iota}_R({}^s(\mathscr{I})) = \{\tilde{Y}, \tilde{\emptyset}, L_R({}^s\mathscr{I}), U_R({}^s\mathscr{I}), B_R({}^s\mathscr{I})\}$ forms a topology on $\tilde{Y}$. We say that $(\tilde{Y}, \mathscr{U}_A, \tilde{\iota}_R({}^s\mathscr{I}))$ is a nano soft ideal topological space over $\tilde{Y}$ with respect to ${}^s(\mathscr{I})$.*

Then the members of $\tilde{\iota}_R({}^s(\mathscr{I}))$ are said to be nano soft ideal open sets in $\tilde{Y}$. Their soft complements are said to be nano soft ideal closed sets.

Definition 2.23. [20] *Let $(\tilde{Y}, \mathscr{U}_A, \tilde{\iota}_R({}^s\mathscr{I}))$ be a nano soft ideal topological space over $\tilde{Y}$ with E as the set of parameters. If $\mathscr{V}_E \subseteq \tilde{Y}$ then the soft union of all nano soft ideal open sets contained in $\mathscr{V}_E$ is called nano soft ideal interior of $\mathscr{V}$, that is $NSIint(\mathscr{V})$. Alternatively (and for brevity), we shall use denotation $\mathscr{V}^o$. The soft intersection of all nano soft ideal closed sets in which $\mathscr{V}$ is contained, is called nano soft ideal closure of $\mathscr{V}$, that is $NSIcl(\mathscr{V})$. Alternatively, we shall write $\mathscr{V}^{cl}$.*

Definition 2.24. [20] *Let $(\tilde{Y}, \mathscr{U}_A, \tilde{\iota}_R({}^s\mathscr{I}))$ be a nano soft ideal topological space over $\tilde{Y}$ with parameter set E. A soft subset $\mathscr{V}_E$ of this space is called nano soft $^s(\mathscr{I})$ pre open if $\mathscr{V}_E \subseteq NSIint(NSIcl(\mathscr{U}_A))$.*

3. On Nano Soft $^s(\mathscr{I})$ Mappings

In this section, we define nano soft $^s(\mathscr{I})$ continuous functions. Their basic properties are discussed with somewhat detailed proofs.

Definition 3.1. *Let $(\tilde{X}, \tilde{\iota}_{1_R}({}^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_R}({}^s(\mathscr{K})))$ be two nano soft ideal topological spaces with the same parameter set E. We define a nano soft ideal function $\chi : \tilde{X} \to \tilde{Y}$ as a function that takes soft points from $\tilde{X}$ as its arguments and returns soft points from $\tilde{Y}$ as values.*

Remark 3.2. *For simplicity we may assume that we restrict our considerations only to those functions which do not change parameters: that is, to those that transform $\tilde{x}_a$ (where $x \in X$, i.e. $\tilde{x}_a \in \tilde{X}$) into $\tilde{y}_a$ (where $y \in Y$, i.e. $\tilde{y}_a \in \tilde{Y}$). Note that we assumed that we have the same parameter set E in both spaces. The image $\chi(\mathscr{U})$ of a soft set $\mathscr{U} \subseteq \tilde{X}$ is defined as the soft union of all the soft points of the form $\chi(\tilde{x}_a)$ in $\tilde{Y}$ $\tilde{x}_a \in \mathscr{U}$. As for the pre-image $\chi^{-1}(\mathscr{V})$ of a soft set $\mathscr{V} \subseteq \tilde{Y}$, it is defined as a soft union of such soft points $\tilde{x}_a$ from $\tilde{X}$ that $\chi(\tilde{x}_a) \in \mathscr{V}$. Moreover, we shall assume that each space is considered with respect to the same set E of parameters. As for the ideals, they can be different in both spaces, without any doubt, so we use symbols $^s(\mathscr{I})$ and $^s(\mathscr{K})$. The same with relations: one is R and another one is represented by S. Finally, if there is no risk of ambiguity, we shall write $\tilde{x}$ instead of $\tilde{x}_a$.*

Definition 3.3. *Let $(\tilde{X}, \tilde{\iota}_{1_R}({}^s(\mathscr{I})))$ be a nano soft ideal topological space. Let $\mathscr{C}$ be $NS({}^s(\mathscr{I}))$-open in this space. If a soft point $\tilde{x}_a \in \mathscr{C}$ then we say that $\mathscr{C}$ is a (nano soft ideal) open neighbourhood of $\tilde{x}_a$.*

Definition 3.4. *Let $(\tilde{X}, \tilde{\iota}_{1_R}({}^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_S}({}^s(\mathscr{K})))$ be two nano soft ideal topological spaces. Then $\chi : \tilde{X} \to \tilde{Y}$ is called as $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ nano soft continuous at a soft point $\tilde{x} \in \tilde{X}$ if corresponding to every $\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ nano soft $^s(\mathscr{I})$ open set $\mathscr{H}$ containing $\chi(\tilde{x})$ there exists a $\tilde{\iota}_{1_R}({}^s(\mathscr{I}))$ nano soft $^s(\mathscr{I})$ open set $\mathscr{V}$ containing $\tilde{x}$ $\chi(\mathscr{V}) \subseteq \mathscr{H}$.*

We say that such a function is $NS({}^s(\mathscr{I}))$ continuous (at soft point $\tilde{x}_a$).

Definition 3.5. *The mapping $\chi : \tilde{X} \to \tilde{Y}$ is said to be $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ nano soft $^s(\mathscr{I})$ continuous if it is nano soft $^s(\mathscr{I})$ continuous for every soft point of $\tilde{X}$.*

Theorem 3.6. *Let $(\tilde{X}, \tilde{\iota}_{1_R}({}^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_S}({}^s(\mathscr{K})))$ be two nano soft ideal TS. Then a function $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ continuous iff*

the inverse image under χ of every $\tilde{\iota}_{2_R}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) open set is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open.

Proof. ($\Rightarrow$)

Suppose that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) continuous at every soft point $\tilde{x}$ of X. Prove that if $\mathscr{H}$ is any arbitrary $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) open set then $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open set.

If $\chi^{-1}(\mathscr{H}) = \tilde{\emptyset}$ then (taking into account the fact that $\tilde{\emptyset}$ is $\tilde{\iota}_R(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open), it follows that $\chi^{-1}(\mathscr{H})$ is NS($^s(\mathscr{I})$) open. But if $\chi^{-1}(\mathscr{H}) \neq \tilde{\emptyset}$ then let $\tilde{x}$ be any arbitrary element of $\chi^{-1}(\mathscr{H})$ so that $\chi(\tilde{x}) \in \mathscr{H}$. As χ is continuous, hence by definition there exists $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open set $\mathscr{V}_{\tilde{x}}$ containing $\tilde{x}$ (and corresponding to $\mathscr{H}$) $\chi(G_{\tilde{x}}) \subseteq \mathscr{H}$ or (equivalently) $G_{\tilde{x}} \subseteq \chi^{-1}(\mathscr{H})$. Hence $\tilde{x} \in G_{\tilde{x}} \subseteq \chi^{-1}(\mathscr{H})$.

It is clear that $\chi^{-1}(\mathscr{H})$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ neighbourhood of $\tilde{x}$ but since $\tilde{x}$ is arbitrary, it follows that $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_R(^s(\mathscr{I}))$ neighbourhood of each of its points. Hence $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open (since every $\tilde{\iota}_R(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$)-open set is a neighbourhood of each of its points).

($\Leftarrow$)

Conversely, assume that the inverse image $\chi^{-1}(\mathscr{H})$ of every $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) open set $\mathscr{H}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open. Prove that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$, $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$)-continuous (that is, it is NS($^s(\mathscr{I})$) continuous at every point $\tilde{x}$ of X).

Let $\mathscr{H}$ be any $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) open set containing $\chi(\tilde{x})$, so that $\tilde{x} \in \chi^{-1}(\mathscr{H})$ where $\chi^{-1}(\mathscr{H})$ is given to be $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) open. Let us put $\chi^{-1}(\mathscr{H}) = \mathscr{V}$ where $\mathscr{V}$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open set containing $\tilde{x}$. Also $\chi(\mathscr{V}) = (\chi^{-1}(\mathscr{H})) \subseteq \mathscr{H}$. Hence, by the definition, χ is NS($^s(\mathscr{I})$) continuous at $\tilde{x}$, but as $\tilde{x}$ is arbitrary, it follows that χ is $(\tilde{X}, \tilde{\iota}_{1_R}(^s(\mathscr{K})))$ continuous at every point $\tilde{x}$ of $\tilde{X}$. Hence χ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) continuous.

Theorem 3.7. *Let $(\tilde{X}, \tilde{\iota}_{1_R}(^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_S}(^s(\mathscr{K})))$ be two nano soft ideal TS. Then a function $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) continuous iff the inverse image under χ of every $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) closed subset of $\tilde{Y}$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) closed subset of $\tilde{X}$.*

Proof. ($\Rightarrow$)

We assume that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) continuous. Hence, it is continuous at every point $\tilde{x}$ of $\tilde{X}$.

Prove that if $\mathscr{H}$ is any $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) closed set, then $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) closed.

Now, assume that $\mathscr{H}$ is $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) closed set. Then $\tilde{Y} \setminus \mathscr{H}$ is

$\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ open. Hence, $\chi^{-1}(\tilde{Y} \setminus \mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ open as χ is NS$(^s(\mathscr{I}))$ continuous. Then $\chi^{-1}(\tilde{Y}) \setminus \chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ open. Then $\tilde{X} \setminus \chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_R(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ open. Therefore $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed.

($\Leftarrow$)

Conversely, assume that the inverse image of every $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ closed set is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed. To prove that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ is NS$(^s(\mathscr{I}))$ continuous. Let $\mathscr{V}$ be any $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ open set so that $\tilde{Y} \setminus \mathscr{V}$ is $\tilde{\iota}_{2_R}(^s(\mathscr{K})$ NS$(^s(\mathscr{I}))$ closed set. Hence by given condition $\chi^{-1}(\tilde{Y} \setminus \mathscr{V})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed set. So $\chi^{-1}(\tilde{Y}) \setminus \chi^{-1}(\mathscr{V})$ is $\tilde{\iota}_{1_R(^s(\mathscr{I}))}$ NS$(^s(\mathscr{I}))$ closed set. Then $\tilde{X} \setminus \chi^{-1}(\mathscr{V})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed set. Therefore $\chi^{-1}(\mathscr{V})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ open set. Above shows that the inverse image of every $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ open set $\mathscr{V}$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ open set and hence χ is NS$(^s(\mathscr{I}))$ continuous.

Theorem 3.8. *If $(\tilde{X}, \tilde{\iota}_{1_R}(^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_S}(^s(\mathscr{K})))$ are two nano soft ideal TS, then $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ continuous iff for every soft subset $\mathscr{A}$ of $\tilde{X}$, $\chi(\mathscr{A}^{cl}) \subseteq (\chi(\mathscr{A}))^{cl}$.*

Proof. ($\Rightarrow$)

Assume that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ continuous. Prove that for any soft subset $\mathscr{A}$ of $\tilde{X}$, the following holds: $\chi(\mathscr{A}^{cl}) \subseteq (\chi(\mathscr{A}))^{cl}$.

The fact that $\mathscr{A} \subseteq \tilde{X}$ implies that $\chi(\mathscr{A}) \subseteq \tilde{Y}$. Hence $(\chi(\mathscr{A}))^{cl}$ is a $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ closed set. Therefore, $\chi^{-1}((\chi(\mathscr{A}))^{cl})$ is NS$(^s(\mathscr{I}))$ closed subset of $\tilde{X}$ (because χ is NS$(^s(\mathscr{I}))$ continuous). Then we recall the natural fact that $\chi(\mathscr{A}) \subseteq (\chi(\mathscr{A}))^{cl}$. From this we infer that $\mathscr{A} \subseteq \chi^{-1}((\chi(\mathscr{A}))^{cl})$. Hence, $\chi^{-1}((\chi(\mathscr{A}))^{cl})$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed set containing $\mathscr{A}$. But at the same time, $\mathscr{A}^{cl}$ is the smallest $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed set containing $\mathscr{A}$. This implies that $\mathscr{A}^{cl} \subseteq \chi^{-1}((\chi(\mathscr{A}))^{cl})$. Hence, $\chi(\mathscr{A}^{cl}) \subseteq (\chi(\mathscr{A}))^{cl}$.

($\Leftarrow$)

Conversely, assume that for every soft subset $\mathscr{A}$ of $\tilde{X}$, $\chi(\mathscr{A}^{cl}) \subseteq (\chi(\mathscr{A}))^{cl}$. Prove that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ continuous.

Let $\mathscr{H}$ be any $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ closed subset of $\tilde{Y}$. In order to show χ to be NS$(^s(\mathscr{I}))$ continuous we shall show that $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed subset of $\tilde{X}$.

Clearly, $\chi^{-1}(\mathscr{H}) \subseteq \tilde{X}$, so $\chi((\chi^{-1}(\mathscr{H}))^{cl}) \subseteq (\chi(\chi^{-1}(\mathscr{H})))^{cl} \subseteq \mathscr{H}^{cl}$. Hence $(\chi^{-1}(\mathscr{H}))^{cl} \subseteq \chi^{-1}(\mathscr{H}^{cl}) = \chi^{-1}(\mathscr{H})$. Thus $(\chi^{-1}(\mathscr{H}))^{cl} \subseteq \chi^{-1}(\mathscr{H})$. However, $\chi^{-1}(\mathscr{H}) \subseteq ((\chi^{-1}(\mathscr{H})))^{cl}$. Therefore, $\chi^{-1}(\mathscr{H})$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS$(^s(\mathscr{I}))$ closed. Hence, our function is $\tilde{\iota}_{1_R}(^s(\mathscr{I})), \tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS$(^s(\mathscr{I}))$ continuous.

Theorem 3.9. *Let $(\tilde{X}, \tilde{\iota}_{1_R}(^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_S}(^s(\mathscr{K})))$ be two nano soft ideal TS,*

then $\chi : \tilde{X} \to \tilde{Y}$ *is* $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ *continuous iff for every soft subset* $\mathscr{C}$ *of* $\tilde{Y}$, $(\chi^{-1}(\mathscr{C}))^{cl} \subseteq \chi^{-1}(\mathscr{C}^{cl})$.

Proof. $(\Rightarrow)$

Assume that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ continuous. Prove that for any soft subset $\mathscr{C}$ of $\tilde{Y}$, $(\chi^{-1}(\mathscr{C}))^{cl} \subseteq \chi^{-1}(\mathscr{C}^{cl})$.

Let $\mathscr{C} \subseteq \tilde{Y}$. Hence $\mathscr{C}^{cl}$ is $\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ closed. As χ is $NS({}^s(\mathscr{I}))$ continuous, therefore $\chi^{-1}(\mathscr{C}^{cl})$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I}))$ $NS({}^s(\mathscr{I}))$ closed subset of $\tilde{X}$. Hence $(\chi^{-1}(\mathscr{C})^{cl})^{cl} = \chi^{-1}(\mathscr{C}^{cl})$. But the fact that $\mathscr{C} \subseteq \mathscr{C}^{cl}$ implies that $\chi^{-1}(\mathscr{C}) \subseteq \chi^{-1}(\mathscr{C}^{cl})$. Then $(\chi^{-1}(\mathscr{C}))^{cl} \subseteq (\chi^{-1}(\mathscr{C})^{cl})^{cl} = \chi^{-1}(\mathscr{C}^{cl})$. Hence we get $(\chi^{-1}(\mathscr{C}))^{cl} \subseteq \chi^{-1}(\mathscr{C}^{cl})$.

$(\Leftarrow)$

Conversely, suppose that for every soft subset $\mathscr{C}$ of $\tilde{Y}$, $(\chi^{-1}(\mathscr{C}))^{cl} \subseteq \chi^{-1}(\mathscr{C}^{cl})$. We would like to prove that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ continuous.

Let $\mathscr{C}$ be any $\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ closed subset of $\tilde{Y}$ so that $\mathscr{C}^{cl} = \mathscr{C}$. Now, by given condition, $(\chi^{-1}(\mathscr{C}))^{cl} \subseteq \chi^{-1}(\mathscr{C}^{cl}) = \chi^{-1}(\mathscr{C})$. But $\chi^{-1}(\mathscr{C}) \subseteq (\chi^{-1}(\mathscr{C}))^{cl}$. Therefore $(\chi^{-1}(\mathscr{C}))^{cl} = \chi^{-1}(\mathscr{C})$. Hence we get that $\chi^{-1}(\mathscr{C})$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I}))$ $NS({}^s(\mathscr{I}))$ closed set. Hence χ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ continuous.

Theorem 3.10. *If* $(\tilde{X}, \tilde{\iota}_{1_R}({}^s(\mathscr{I})))$ *and* $(\tilde{Y}, \tilde{\iota}_{2_S}({}^s(\mathscr{K})))$ *are two nano soft ideal TS. Then a function* $\chi : \tilde{X} \to \tilde{Y}$ *is* $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ *continuous iff for every soft subset* $\mathscr{C}$ *of* $\tilde{Y}$, $(\chi^{-1}(\mathscr{C}))^{o} \supseteq \chi^{-1}(\mathscr{C}^{o})$.

Proof. $(\Rightarrow)$

Assume that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I}))\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ continuous. Prove that for any soft subset $\mathscr{C}$ of $\tilde{Y}$, $(\chi^{-1}(\mathscr{C}))^{o} \supseteq \chi^{-1}(\mathscr{C}^{o})$.

Let $\mathscr{C}$ be any subset of $\tilde{Y}$. Then $\mathscr{C}^{o}$ is $\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ open set. As we know, χ is $NS({}^s(\mathscr{I}))$ continuous. It follows that $\chi^{-1}(B^{o})$ is a $\tilde{\iota}_{1_R}({}^s(\mathscr{I}))$ $NS({}^s(\mathscr{I}))$ open subset of $\tilde{X}$. Therefore $(\chi^{-1}(B^{o}))^{o} = \chi^{-1}(B^{o})$. Again, $B^{o} \subseteq B$ implies that $\chi^{-1}(\mathscr{C}^{o}) \subseteq \chi^{-1}(\mathscr{C})$. Then $(\chi^{-1}(\mathscr{C}^{o}))^{o} \subseteq (\chi^{-1}(\mathscr{C}))^{o}$. This implies that $\chi^{-1}(\mathscr{C}^{o}) \subseteq (\chi^{-1}(\mathscr{C}))^{o}$. Then $(\chi^{-1}(\mathscr{C}))^{o} \supset \chi^{-1}(\mathscr{C}^{o})$.

$(\Leftarrow)$

Suppose that for every soft subset $\mathscr{C}$ of $\tilde{Y}$, $(\chi^{-1}(\mathscr{C}))^{o} \supseteq \chi^{-1}(\mathscr{C}^{o})$. Prove that $\chi : \tilde{X} \to \tilde{Y}$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ is $NS({}^s(\mathscr{I}))$ continuous.

Let $\mathscr{C}$ be any $\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ open subset of $\tilde{Y}$ so that $\mathscr{C}^{o} = \mathscr{C}$. Now $(\chi^{-1}(\mathscr{C}))^{o} \supseteq \chi^{-1}(\mathscr{C}^{o}) = \chi^{-1}(\mathscr{C})$. Then $(\chi^{-1}(\mathscr{C}))^{o} \supseteq \chi^{-1}(\mathscr{C})$. But $(\chi^{-1}(\mathscr{C}))^{o} \subseteq \chi^{-1}(\mathscr{C})$. Hence we get $(\chi^{-1}(\mathscr{C}))^{o} = \chi^{-1}(\mathscr{C})$. But we know that $\mathscr{A}^{o} = \mathscr{A}$ iff $\mathscr{A}$ is $NS({}^s(\mathscr{I}))$ open for any $\mathscr{A}$. Hence we get that $\chi^{-1}(\mathscr{C})$ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I}))$ $NS({}^s(\mathscr{I}))$ open where $\mathscr{C}$ is $\tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ open. Therefore χ is $\tilde{\iota}_{1_R}({}^s(\mathscr{I})), \tilde{\iota}_{2_S}({}^s(\mathscr{K}))$ $NS({}^s(\mathscr{I}))$ continuous.

Using similar methods, we are able to prove the following theorem which typical for TS and holds also in our nano soft ideal environment. Strictly speaking, this would require some new definitions (e.g. definitions of base and subbase). However, they would appear in a very natural way.

Theorem 3.11. *Let $(\tilde{X}, \tilde{\iota}_{1_R}(^s(\mathscr{I})))$ and $(\tilde{Y}, \tilde{\iota}_{2_S}(^s(\mathscr{K})))$ be two nano soft ideal TS. Assume that we have a function $\chi : \tilde{X} \to \tilde{Y}$. Then the following statements are equivalent.*

(i) *The function χ is NS($^s(\mathscr{I})$) continuous.*

(ii) *The inverse image under χ of $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) open set is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open.*

(iii) *The inverse image under χ of every $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ NS($^s(\mathscr{I})$) closed set is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) closed.*

(iv) *The inverse image under χ of every $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ neighbourhood of $\chi(x)$ is $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ neighbourhood of x.*

(v) *The inverse image under χ of every member of base B for $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open subset of X.*

(vi) *The inverse image under χ of every member of sub-base B for $\tilde{\iota}_{2_S}(^s(\mathscr{K}))$ is a $\tilde{\iota}_{1_R}(^s(\mathscr{I}))$ NS($^s(\mathscr{I})$) open subset of X.*

(vii) *$\chi(\mathscr{A}^{cl}) \subseteq [\chi(\mathscr{A})]^{cl}$ for every soft subset $\mathscr{A}$ of $\tilde{X}$.*

(viii) *Let $[\chi^{-1}(\mathscr{C}l)]^{cl} \subseteq \chi^{-1}(\mathscr{C}^{cl})$ for every soft subset $\mathscr{C}$ of $\tilde{Y}$.*

(ix) *Let $[\chi^{-1}(\mathscr{C})]^o \subseteq \chi^{-1}(\mathscr{C}^o)$ for every $\mathscr{C} \subseteq \tilde{Y}$.*

4. Conclusion

This paper introduces the concept of nano soft $^s(\mathscr{I})$ continuous functions based on the idea of nano soft ideal topological space. Moreover, we defined nano soft $^s(\mathscr{I})$-open sets and their properties were discussed. The whole project can be continued. For example, we see that it is possible to generalize these results: instead of ideals we could use hereditary classes or any other specific class of families of classical sets. We can drop the assumption that our function does not change parameters (as we assumed in Remark 3.2). This would require cooperation of two mappings: one for the elements of underlying universes and the second one for the parameters. This is a common approach in soft sets theory.

Without any doubt, many "soft" results in topology are similar to those that can be obtained in classical setting. However, they are more general. Moreover, some subtle differences can appear. For example, we know that there are many non-equivalent understandings of union, intersection and complement in soft setting. In some cases, de Morgan laws (or other typical algebraic laws) are not satisfied.

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